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ON THE WIDTHS OF FINITE DISTRIBUTIVE LATTICES*

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The following conjecture of U Faigle and B Sands is proved: For every number $R > 0$ there exists a number $n(R)$ such that if $\mathscr{L}$ is a finite distributive lattice whose width $w(\mathscr{L})$ (size of the largest antichain) is at least $n(R)$, then $|\mathscr{L}| \geq Rw(\mathscr{L})$. In words this says that as one considers increasingly large distributive lattices, the maximum sized antichain contains a vanishingly small proportion of the elements.

1. Introduction

The width $w(\cdot)$ of a partially ordered set is the cardinality of its largest antichain. The purpose of this paper is to prove the following result:

Theorem 1.1. *For every number $R > 0$ there exists a number $n(R)$ such that if $\mathscr{L}$ is a finite distributive lattice with $w(\mathscr{L}) \geq n(R)$, then $|\mathscr{L}| \geq Rw(\mathscr{L})$. Symbolically*

$$\lim_{|\mathscr{L}| \to \infty} \frac{w(\mathscr{L})}{|\mathscr{L}|} = 0.$$

This was conjectured by Faigle and Sands [7] who proved

$$\lim_{|\mathscr{L}| \to \infty} \frac{w(\mathscr{L})}{|\mathscr{L}|} \leq \tfrac{1}{3}.$$

In fact, our results show that for any distributive lattice $\mathscr{L}$, $w(\mathscr{L})/|\mathscr{L}| = O((\log\log|\mathscr{L}|)^{-(1/2-\varepsilon)})$ for any $\varepsilon > 0$ (see Section 7). We believe that this is far from best possible and that $w(\mathscr{L})/|\mathscr{L}| = O((\log|\mathscr{L}|)^{-1/2})$. The finite Boolean lattices show that this bound would be best possible.

Since every finite distributive lattice is isomorphic to a sublattice of the lattice of subsets of some finite set, and vice versa, the theorem can be interpreted in terms of extremal set theory. Let $\mathscr{S}$ be a collection of finite sets and $G(\mathscr{S})$ the collection obtained by closing $\mathscr{S}$ under union and intersection. In this context Theorem 1.1 becomes

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Theorem 1.2. *For every number $R > 0$ there exists a number $n(R)$ such that if $\mathcal{S}$ is an antichain of finite sets with $|\mathcal{S}| \geq n(R)$, then $G(\mathcal{S}) \geq R\,|\mathcal{S}|$.*

2. Notation and basic facts

We provide a quick review of properties of distributive lattices that will be needed in the proof of Theorem 1.1. For a more complete discussion see [1–3].

Let P be a finite partially ordered set. We denote elements of P by lower case letters and subsets of P by capital letters. An *order ideal* I of P is a subset of P that is closed downward, i.e., $x \in I$ and $y \leq x$ imply $y \in I$. The set of order ideals of P, ordered by inclusion, is a sublattice of the lattice of all subsets of P and is therefore distributive. Now, there is a natural bijection between the ideals of P and the antichains of P under which each ideal I corresponds to the set of elements that are maximal in I. Using this correspondence, the set of antichains of P inherits a distributive lattice structure. The inherited order on antichains is $A_1 \leq A_2$ if for every $x \in A_1$ there exists $y \in A_2$ such that $x \leq y$. The set of antichains of P together with this ordering is denoted $\mathcal{L}(P)$. For $A_1, A_2 \in \mathcal{L}(P)$, $A_1 \vee A_2$ is the set of elements maximal in $A_1 \cup A_2$ and hence $A_1 \vee A_2 \subseteq A_1 \cup A_2$.

The fundamental fact about finite distributive lattices is that the map from partially ordered sets to distributive lattices is invertible.

Lemma 2.1 (Birkhoff representation theorem [2]). *Let $\mathcal{L}$ be a finite distributive lattice. Let $P(\mathcal{L})$ be the set of join irreducible elements of $\mathcal{L}$, i.e., elements that cover exactly one element. Then $\mathcal{L}$ is isomorphic to the lattice of antichains of P.*

Throughout this paper $\mathcal{L}$ denotes a finite distributive lattice. We will often make explicit and implicit use of the above representation theorem, viewing $\mathcal{L}$ as the set of antichains of a finite partially ordered set P. Elements of $\mathcal{L}$ are denoted by upper case latters A, B, X, Y and Z. Script letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$ and $\mathcal{N}$ are used to denote subsets of $\mathcal{L}$.

The *height* of an element $x \in \mathcal{L}$, written $h(X)$, is the cardinality of the largest chain of elements strictly less than X. The ith rank of $\mathcal{L}$, $\mathcal{L}[i]$ is the set of elements of height i. For i negative or i greater than the maximum height of an element of $\mathcal{L}$ we take $\mathcal{L}[i]$ to be empty. It is well known that distributive lattices are *graded*, i.e., if $X \in \mathcal{L}[i]$ and X covers Y, then $Y \in \mathcal{L}[i-1]$. A *strip* $\mathcal{L}[j, k]$ of $\mathcal{L}$ is a union of consecutive ranks, that is, for $j < k$, $\mathcal{L}[j, k]$ is equal to $\bigcup_{i=j}^{k} L[i]$. More generally, if $\mathcal{A}$ is any subset of $\mathcal{L}$, we write $\mathcal{A}[j, k]$ for $\mathcal{A} \cap \mathcal{L}[j, k]$.

Let $X \in \mathcal{L}$. For an integer $i \geq 0$, $D_i(X)$ is the set of elements Y such that $Y \leq X$ and $h(Y) = h(X) - i$. The cardinality of $D_i(X)$ is denoted $d_i(X)$. In particular $d_1(X)$ is the number of elements of $\mathcal{L}$ that are covered by X and is called the *down degree* of X. Using Lemma 2.1, it is easy to show that the down degree of X

in $\mathscr{L}$ is equal to the cardinality of X viewed as an antichain of $P(\mathscr{L})$, i.e., $d_1(X) = |X|$.

Similarly, we define $U_i(X)$ to be the set of elements Y such that $Y \geqslant X$ and $h(Y) = h(X) + i$, and define $u_i(X) = |U_i(X)|$. The quantity $u_1(X)$ is the number of elements of $\mathscr{L}$ that cover X and is called the *up degree* of X.

Lemma 2.2. *Let $X, Y \in \mathscr{L}$ with $X < Y$ and $h(Y) - h(X) = j$. Then*

(i) $d_1(Y) \leqslant d_1(X) + j$,
(ii) $u_1(X) \leqslant u_1(Y) + j$.

Proof. We prove (i); (ii) follows by duality. Using Lemma 2.1, we view X and Y as antichains of $P(\mathscr{L})$. The conditions $X < Y$ and $h(Y) - h(X) = j$ imply that the order ideals $I(X)$ and $I(Y)$ of $P(\mathscr{L})$ generated by X and Y satisfy $I(X) \subseteq I(Y)$ and $|I(Y) - I(X)| = j$. Since $Y \subseteq X \cup (I(Y) - I(X))$, we have $|Y| \leqslant |X| + j$ and hence $d_1(Y) \leqslant d_1(X) + j$. □

Let $J_i(X)$ (respectively $M_i(X)$) denote the set of elements that can be expressed as the join (resp. meet) of exactly i elements that cover (resp. are covered by) X. An easy consequence of the Birkhoff representation theorem is

Lemma 2.3. *For any $X \in \mathscr{L}$ and $i \geqslant 0$,*

(i) $J_i(X) \subseteq U_i(X)$ *and* $M_i(X) \subseteq D_i(X)$,
(ii) $Y \in J_i(X)$ *if and only if* $X \in M_i(Y)$,
(iii) $|J_i(X)| = \binom{u_1(X)}{i}$ *and* $|M_i(X)| = \binom{d_1(X)}{i}$.

3. The structure of the proof

The proof of Theorem 1.1 is based on two lemmas. Actually, Lemma 3.2 contains the heart of the matter. It turns out that the main line of argument, which is essentially the proof of Lemma 3.2, requires the technical assumption that the elements of our antichain have large down degrees. The case that this assumption fails is treated separately in Lemma 3.1.

Lemma 3.1. *Let P be an ordered set and $\mathscr{A}$ an antichain of $\mathscr{L} = \mathscr{L}(P)$ such that each member of $\mathscr{A}$ has size at most b. Then*

$$|\mathscr{L}(P)| \geqslant c(b)\, |\mathscr{A}|^{1+\varepsilon(b)},$$

where $\varepsilon(i) = 1/(2^i - 1)$ and $c(b) = \prod_{i=1}^{b} 1/(1 + \varepsilon(i))$ (which is a decreasing function of b ranging between 0.5 *and* 0.288).

The Birkhoff representation theorem allows us to restate Lemma 3.1 as

Lemma 3.1′. *Let $\mathscr{L}$ be a distributive lattice and $\mathscr{A}$ be an antichain of $\mathscr{L}$ all of whose elements have down degree at most b. Then*

$$|\mathscr{L}| \geq c(b)\,|\mathscr{A}|^{1+\varepsilon(b)}.$$

Lemma 3.2. *Let R' be a positive number. There exist integers t and b such that if $\mathscr{L}$ is a distributive lattice, q is any index, and $\mathscr{A}$ is an antichain contained in $\mathscr{L}[q+t+1, q+3t]$ in which every element has down degree at least b, then*

$$|\mathscr{L}[q+1, q+4t]| \geq R'\,|\mathscr{A}|.$$

To prove Theorem 1.1 from Lemmas 3.1 and 3.2, let R be any positive integer. We show that there is an integer $n(R)$ so that if $\mathscr{A}$ is an antichain of $\mathscr{L}$ and $|\mathscr{A}| \geq n(R)$, then $|\mathscr{L}| \geq R\,|\mathscr{A}|$. Set $R' = 3R$ and let t and b be the integers given by Lemma 3.2. Let $n(R) = 3(R'/c(b))^{1/\varepsilon(b)}$, where $c(b)$ and $\varepsilon(b)$ are as in Lemma 3.1. Assume $|\mathscr{A}| \geq n(R)$ and partition $\mathscr{A} = \mathscr{A}_b \dot{\cup} \mathscr{A}^b$, where $\mathscr{A}_b$ is the set of elements of down degree at most b. If $|\mathscr{A}_b| \geq \frac{1}{3}|\mathscr{A}|$, then by Lemma 3.1′,

$$\begin{aligned}|\mathscr{L}| &\geq c(b)\,|\mathscr{A}_b|^{1+\varepsilon(b)} \geq \frac{|\mathscr{A}|}{3} c(b) \left(\frac{n(R)}{3}\right)^{\varepsilon(b)} \\ &\geq \frac{|\mathscr{A}|}{3} c(b) \frac{R'}{c(b)} = |\mathscr{A}|\,R\end{aligned}$$

as required. So assume $|\mathscr{A}^b| > \frac{2}{3}|\mathscr{A}|$. For $j \geq 0$ let A_j^b be the part of $\mathscr{A}^b$ that belongs to the strip $\mathscr{L}[2jt+1, 2(j+1)t]$ (so the $\mathscr{A}_j^b$ partition $\mathscr{A}^b$). Setting $q = (2j-1)t$ in Lemma 3.2, we have for each integer j

$$|\mathscr{L}[(2j-1)t+1, (2j+3)t]| > R'\,|\mathscr{A}_j^b|.$$

Now each rank of $\mathscr{L}$ belongs to $\mathscr{L}[(2j-1)t+1, (2j+3)t]$ for exactly two values of j, so summing on j we get $2|\mathscr{L}| > R'\,|\mathscr{A}^b|$ which implies $|\mathscr{L}| > R\,|\mathscr{A}|$. □

In Section 4, we prove Lemma 3.1 by a straightforward induction. The proof of Lemma 3.2 requires another lemma, presented in Section 5. This lemma says that if $\mathscr{M}$ is a subset of a rank of $\mathscr{L}$, then $u_j(\mathscr{M}) \geq (|\mathscr{M}|/d_1(\mathscr{M}))^j\,|\mathscr{M}|\,(1-\theta)$, where θ depends on j and the minimum down degree of $\mathscr{M}$ and tends to 0 as the minimum down degree gets large. Section 6 contains the proof of Lemma 3.2. Finally, various open questions are mentioned in Section 7.

4. Proof of Lemma 3.1

We proceed by induction on $|P|$. For $|P| = 1$, the result is trivial. Let t be the maximum number of members of $\mathscr{A}$ having an element in common.

Lemma 4.1. *For $B \in \mathscr{L}$, the number of pairs $(A_1, A_2) \in \mathscr{A} \times \mathscr{A}$ such that $B = A_1 \vee A_2$ is at most $2t^2$.*

Proof. If $B \in \mathscr{A}$, then (B, B) is the only such pair and if $B \subset A$ for some $A \in \mathscr{A}$ then there are no such pairs, so assume $B \not\subseteq A$ for any $A \in \mathscr{A}$. Recall $B = A_1 \vee A_2$ implies $B \subseteq A_1 \cup A_2$. Fix $y \in B$. In any such pair (A_1, A_2) either $y \in A_1$, or $y \in A_2$ and there are at most t sets in A containing y. Having chosen a set A containing y, there are at most t ways to select a set A' such that $A \vee A' = B$ (all such sets contain $B - A$), and two ways to order A and A', so the lemma follows. □

An immediate consequence of the lemma is that the set of joins $A_1 \vee A_2$ for $(A_1, A_2) \in \mathscr{A} \times \mathscr{A}$ has at least $|\mathscr{A}|^2/2t^2$ distinct members so $|\mathscr{L}| \geq |\mathscr{A}|^2/2t^2$. If $t \leq |\mathscr{A}|^{(1-\varepsilon(b))/2}$ then Lemma 3.1 follows; so assume

$$t \geq |\mathscr{A}|^{(1-\varepsilon(b))/2}, \tag{4.1}$$

and let x be an element of P that belongs to t members of $\mathscr{A}$. Write $P - x$ for the poset obtained by deleting x from P and P/x for the poset obtained by deleting x and all elements comparable to it. Antichains of P/x are in one to one correspondence with antichains of P that contain x, via the map $A \leftrightarrow A \cup \{x\}$. Thus

$$|\mathscr{L}| = |\mathscr{L}(P - x)| + |\mathscr{L}(P/x)|.$$

Write $\mathscr{A}(P - x)$ for $\mathscr{A} \cap \mathscr{L}(P - x)$ and $\mathscr{A}(P/x)$ for the antichain of $\mathscr{L}(P/x)$ consisting of sets of the form $A - x$ for all $A \in \mathscr{A}$ such that $x \in A$. Then

$$|\mathscr{A}| = |\mathscr{A}(P - x)| + |\mathscr{A}(P/x)|.$$

Note that $\mathscr{A}(P/x)$ has size t and consists of antichains of size at most $b - 1$, and $\mathscr{A}(P - x)$ has size $|\mathscr{A}| - t$. By induction

$$|\mathscr{L}(P/x)| \geq c(b-1)t^{1+\varepsilon(b-1)}$$

and

$$|\mathscr{L}(P - x)| \geq c(b)(|\mathscr{A}| - t)^{1+\varepsilon(b)}.$$

Thus

$$\begin{aligned}
|\mathscr{L}| &\geq c(b-1)t^{1+\varepsilon(b-1)} + c(b)(|\mathscr{A}| - t)^{1+\varepsilon(b)} \\
&= c(b)\,|\mathscr{A}|^{1+\varepsilon(b)}\left(\frac{c(b-1)}{c(b)}\,\frac{t^{1+\varepsilon(b-1)}}{|\mathscr{A}|^{1+\varepsilon(b)}} + \left(\frac{|\mathscr{A}| - t}{|\mathscr{A}|}\right)^{1+\varepsilon(b)}\right) \\
&\geq c(b)\,|\mathscr{A}|^{1+\varepsilon(b)}\left(\frac{c(b-1)}{c(b)}\,\frac{t^{1+\varepsilon(b-1)}}{|\mathscr{A}|^{1+\varepsilon(b)}} + 1 - (1+\varepsilon(b))\frac{t}{|\mathscr{A}|}\right).
\end{aligned}$$

Now it is enough to show that the expression in parentheses is at least 1, which follows if we show

$$\frac{c(b-1)}{c(b)}\,\frac{t^{1+\varepsilon(b-1)}}{|\mathscr{A}|^{1+\varepsilon(b)}} \geq (1+\varepsilon(b))\frac{t}{|\mathscr{A}|}.$$

Since $c(b-1)/c(b) = 1 + \varepsilon(b)$ this is equivalent to $t \geq |\mathscr{A}|^{\varepsilon(b)/\varepsilon(b-1)}$. For $\varepsilon(b) = 1/(2^b - 1)$ this follows from the assumption (4.1) that $t \geq |\mathscr{A}|^{(1-\varepsilon(b))/2}$. $\square$

5. An inequality for $|\mathcal{M}|$, $d_1(\mathcal{M})$ and $u_j(\mathcal{M})$

In this section, we present a lemma that formalizes and extends the following intuition about distributive lattices: if $\mathcal{M}$ is a subset of a rank of $\mathscr{L}$, then $d_1(\mathcal{M})$ and $u_1(\mathcal{M})$ cannot both be small relative to $|\mathcal{M}|$.

Lemma 5.1. *Let $\mathscr{L}$ be a distributive lattice and $\mathcal{M}$ a subset of some rank of $\mathscr{L}$, such that $d_1(\mathcal{M}) \leq |\mathcal{M}|$. For any integer $j \geq 1$,*

$$u_j(\mathcal{M}) \geq \left(\frac{|\mathcal{M}|}{d_1(\mathcal{M})}\right)^j |\mathcal{M}| \,(1-\theta),$$

where θ depends only on j and the minimum down degree, b, of elements of $\mathcal{M}$. Furthermore for $b > j$,

$$\theta(b, j) \leq Cj^2 \ln(j+1)/(b-j)$$

for some constant C and thus for any fixed j, θ tends to 0 as b gets large.

The proof requires a technical lemma.

Lemma 5.2. *Let S be a finite set and $\alpha_1, \alpha_2, \ldots, \alpha_k$ be functions from S to the positive reals. Then*

$$\prod_{i=1}^{k}\left(\sum_{s\in S}\alpha_i(s)\right) \geq \prod_{s\in S}\left(\prod_{i=1}^{k}\alpha_i(s)\right)^{1/|S|}|S|^k.$$

Proof. We have

$$\prod_{i=1}^{k}\left(\sum_{s\in S}\alpha_i(s)\right) = \prod_{i=1}^{k}\left[\frac{\sum_{s\in S}\alpha_i(s)}{|S|}\right]\cdot|S|^k \geq \prod_{i=1}^{k}\left(\prod_{s\in S}\alpha_i(s)\right)^{1/|S|}\cdot|S|^k$$
$$= \prod_{s\in S}\left(\prod_{i=1}^{k}\alpha_i(s)\right)^{1/|S|}\cdot|S|^k,$$

where the inequality follows from the arithmetic-geometric mean inequality. $\square$

Proof of Lemma 5.1. Let $Y \in U_j(\mathcal{M})$. By lemma 2.3, $|M_j(Y)| = \binom{d_1(Y)}{j}$ and thus

$$\sum_{X\in\mathcal{M}\cap M_j(Y)}\binom{d_1(Y)}{j}^{-1} \leq 1.$$

Therefore

$$u_j(\mathcal{M}) = \sum_{Y \in U_j(\mathcal{M})} 1 \geqslant \sum_{Y \in U_j(\mathcal{M})} \sum_{X \in \mathcal{M} \cap M_j(Y)} \binom{d_1(Y)}{j}^{-1}$$
$$\geqslant \sum_{X \in \mathcal{M}} \sum_{Y \in J_j(X)} \binom{d_1(Y)}{j}^{-1}.$$

Similarly,

$$d_1(\mathcal{M}) \geqslant \sum_{X \in \mathcal{M}} \sum_{Y \in D_1(X)} \frac{1}{u_1(Y)}.$$

Defining, for $X \in \mathcal{M}$,

$$f(X) = \sum_{Y \in J_j(X)} \binom{d_1(Y)}{j}^{-1}, \qquad g(X) = \sum_{Y \in D_1(X)} \frac{1}{u_1(Y)},$$

we obtain, via Lemma 5.2, that for $\mathcal{N} \subseteq \mathcal{M}$,

$$u_j(\mathcal{M}) d_1(\mathcal{M})^j \geqslant \Big(\sum_{X \in \mathcal{N}} f(X)\Big)\Big(\sum_{X \in \mathcal{N}} g(X)\Big)^j \geqslant \prod_{X \in \mathcal{N}} (f(X) g(X)^j)^{1/|\mathcal{N}|} \, |\mathcal{N}|^{j+1}. \tag{5.1}$$

Define $\nu(X)$ to be the maximum up degree of any element in $D_1(X)$. For each $X \in \mathcal{M}$, $g(X) \geqslant d_1(X)/\nu(X)$. By Lemma 2.2, $u_1(X) \geqslant \nu(X) - 1$ and for $Y \in J_j(X)$, $d_1(Y) \leqslant d_1(X) + j$ so

$$f(X) \geqslant \frac{|J_j(X)|}{\binom{d_1(X)+j}{j}} \geqslant \frac{\binom{\nu(X)-1}{j}}{\binom{d_1(X)+j}{j}} = \frac{(\nu(X)-1)_j}{(d_1(X)+j)_j},$$

where $(a)_j = a(a-1)\cdots(a-j+1)$. Hence for each $X \in \mathcal{M}$

$$f(X) g(X)^j \geqslant \frac{d_1(X)^j}{(d_1(X)+j)_j} \frac{(\nu(X)-1)_j}{\nu(X)^j}. \tag{5.2}$$

Combining (5.1) and (5.2) we get that for any $\mathcal{N} \subseteq \mathcal{M}$

$$u_j(\mathcal{M}) \geqslant \frac{|\mathcal{M}|^j}{d_1(\mathcal{M})^j} |\mathcal{M}| \left[\left(\frac{|\mathcal{N}|}{|\mathcal{M}|}\right)^{j+1} \left(\prod_{X \in \mathcal{N}} \frac{d_1(X)^j}{(d_1(X)+j)_j} \frac{(\nu(X)-1)_j}{\nu(X)^j} \right)^{1/|\mathcal{N}|} \right]. \tag{5.3}$$

We want for an appropriately chosen $\mathcal{N}$ to determine a bound on the bracketed term as a function of j and b. First of all, by definition of b, $d_1(X) \geqslant b$ for all $X \in \mathcal{N}$, so $d_1(X)^j/(d_1(X)+j)_j \geqslant b^j/(b+j)_j$ and therefore

$$\prod_{X \in \mathcal{N}} \left(\frac{d_1(X)^j}{(d_1(X)+j)_j} \right)^{1/|\mathcal{N}|} \geqslant \frac{b^j}{(b+j)_j}. \tag{5.4}$$

Now for $i \geqslant 1$, let $\mathcal{N}_i = \{X \in \mathcal{M} \mid \nu(X) = i\}$ and $\alpha_i = |\mathcal{N}_i|/|\mathcal{M}|$.

Let $\mathcal{N} = \bigcup_{i \geqslant j+1} \mathcal{N}_i$. Then

$$\left(\frac{|\mathcal{N}|}{|\mathcal{M}|}\right)^{j+1} \prod_{X \in \mathcal{N}} \left(\frac{(\nu(X)-1)_j}{\nu(X)^j}\right)^{1/|\mathcal{N}|}$$

$$= (1-(\alpha_1+\alpha_2+\cdots+\alpha_j))^{j+1} \prod_{i \geqslant j+1} \left(\frac{(i-1)_j}{i^j}\right)^{\alpha_i/1-(\alpha_1+\cdots+\alpha_j)}$$

$$\geqslant (1-(\alpha_1+\alpha_2+\cdots+\alpha_j))^{j+1} \left(\prod_{i \geqslant j+1} \left(\frac{i-j}{i}\right)^{\alpha_i}\right)^{j/1-(\alpha_1+\alpha_2+\cdots+\alpha_j)}. \tag{5.5}$$

Now,

$$|\mathcal{M}| \geqslant |D_1(\mathcal{M})| = \sum_{Y \in D_1(\mathcal{M})} \sum_{X \in U_1(Y)} \frac{1}{u_1(Y)} \geqslant \sum_{Y \in D_1(\mathcal{M})} \sum_{X \in U_1(Y)} \frac{1}{\nu(X)}$$

$$\geqslant \sum_{X \in \mathcal{M}} \frac{d_1(X)}{\nu(X)} \geqslant b \sum_{X \in \mathcal{M}} \frac{1}{\nu(X)} = b \sum_{i \geqslant 1} \frac{\alpha_i\,|\mathcal{M}|}{i},$$

so that

$$\sum_{i \geqslant 1} \frac{\alpha_i}{i} \leqslant \frac{1}{b}. \tag{5.6}$$

Thus $\alpha_1 + \alpha_2 + \cdots + \alpha_j \leqslant j/b$. Now

$$\prod_{i \geqslant j+1} \left(\frac{i-j}{i}\right)^{\alpha_i} = \exp\left[\sum_{i \geqslant j+1} \alpha_i \ln\left(1-\frac{j}{i}\right)\right].$$

Subject to (5.6), $\sum_{i \geqslant j+1} \alpha_i \ln(1-j/i)$ is minimized by $\alpha_{j+1} = (j+1)/b$ and $\alpha_i = 0$ for $i > j+1$. Thus

$$\prod_{i \geqslant j+1} \left(\frac{i-j}{i}\right)^{\alpha_i} \geqslant \exp[-(j+1)\ln(j+1)/b]. \tag{5.7}$$

Combining (5.4), (5.5) and (5.7), the bracketed term of (5.3) is at least

$$\left(\frac{b-j}{b}\right)^{j+1} \frac{b^j}{(b+j)_j} \exp[-(j+1)j\ln(j+1)/(b-j)]$$

$$\geqslant \frac{(b-j)^{j+1}}{(b+j)_{j+1}} (1-(j+1)j\ln(j+1)/(b-j))$$

$$\geqslant \left(1-\frac{2j(j+1)}{b+j}\right)(1-(j+1)j\ln(j+1)/(b-j))$$

$$\geqslant 1 - Cj^2\ln(j+1)/(b-j). \quad \square$$

6. Proof of Lemma 3.2

As stated previously, this is the central argument of the proof of Theorem 1.1. Before giving the proof, which is rather technical, we give a qualitative sketch. The lemma asserts that if we take any strip $\mathcal{S}$ of $4t$ consecutive ranks (where t is large enough), then any antichain $\mathcal{A}$ that lies in the middle $2t$ ranks of that strip and consists only of elements of sufficiently large down degree contains only a small fraction of the elements of $\mathcal{S}$. Ideally, for each non-empty $\mathcal{A}[i]$, the sets $D_1(\mathcal{A}[i])$, $D_2(\mathcal{A}[i]), \ldots, D_t(\mathcal{A}[i])$ would each contribute about $|\mathcal{A}[i]|$ to the size of $\mathcal{S}$ and these contributions would be disjoint for different values of i (a situation corresponding roughly to case 2 below) thus ensuring that $|\mathcal{S}|$ is a large multiple of $|\mathcal{A}|$.

Unfortunately, two things may happen to prevent this: first the sizes of the $D_j(\mathcal{A}[i])$ may shrink rapidly as j increases and, second, the contributions descended from different $\mathcal{A}[i]$ may overlap. What the proof does is to show that if either of these events (suitably quantified) occurs, then there must be some i in the first $3t$ levels of the strip and a large subset $\mathcal{M}$ of $\mathcal{L}[i]$ satisfying: (1) $\mathcal{M}$ has large down degree, and (2) $d_1(\mathcal{M}) < |\mathcal{M}|(1-\delta)$ for a suitable δ. In this case, the exponential blowup predicted by Lemma 5.1 guarantees that for some k, $u_k(\mathcal{M})$ (which lies in $\mathcal{S}$) and is itself large enough to guarantee that $|\mathcal{S}|$ is a large multiple of $|\mathcal{A}|$.

Proceeding with the proof, we first define the numbers t and b for which the conclusion of the lemma is valid. Motivation for these definitions will be given in the proof. Choose $\delta < 1/2R'$ and let t be an integer satisfying:

$$\sum_{j=0}^{t} (1-\delta)^j > 2R', \tag{6.1}$$

$$8R't(1-\delta)^{t+1} < 1. \tag{6.2}$$

Since

$$\lim_{t\to\infty} \sum_{j=0}^{t} (1-\delta)^j \to \frac{1}{\delta} > 2R' \quad \text{and} \quad \lim_{t\to\infty} t(1-\delta)^t \to 0,$$

any sufficiently large t will work. Let b be an integer large enough to make $\theta(b-t, 2t) \leqslant \frac{1}{2}$, where θ is the function given by Lemma 5.1.

Let q be an integer and $\mathcal{A}$ be an antichain of $\mathcal{L}[q+t+1, q+3t]$ for which every element has down degree at least b. Define $\mathcal{B}$ to be the subset of $\mathcal{A}$ consisting of all ranks of $\mathcal{A}$ having size at least $|\mathcal{A}|/4t$; then $\mathcal{A}-\mathcal{B}$ has at most $2t$ $(|\mathcal{A}|/4t) = \frac{1}{2}|\mathcal{A}|$ elements, so $|\mathcal{B}| \geqslant \frac{1}{2}|\mathcal{A}|$.

We distinguish two cases.

Case 1. For some j with $0 \leqslant j \leqslant 3t-1$ there is a subset $\mathcal{M} \subseteq \mathcal{L}[q+3t-j]$ of size at least $(1-\delta)^{\min(j,t)-1}|\mathcal{A}|/4t$ all of whose elements have down degree at least $b-t$ and such that $d_1(\mathcal{M}) \leqslant (1-\delta)|\mathcal{M}|$. By Lemma 5.1,

$$\begin{aligned} u_{t+\min(j,t)}(\mathcal{M}) &\geqslant \left(\frac{|\mathcal{M}|}{d_1(\mathcal{M})}\right)^{t+\min(j,t)} |\mathcal{M}|(1-\theta(b-t, 2t)) \\ &\geqslant \frac{1}{(1-\delta)^{t+\min(j,t)}} \frac{|\mathcal{A}|}{4t} (1-\delta)^{\min(j,t)-1}\left(\frac{1}{2}\right) \\ &\geqslant \frac{|\mathcal{A}|}{8t(1-\delta)^{t+1}} > |\mathcal{A}|R' \end{aligned}$$

by (6.2). Now since $\mathcal{M} \subseteq \mathcal{L}[q+3t-j]$, $U_{t+\min(j,t)}(\mathcal{M}) \subset \mathcal{L}[q+1, q+4t]$ so $|\mathcal{L}[q+1, q+4t]| \geq |\mathcal{A}|R'$.

Case 2. The negation of Case 1. For h between 1 and t let

$$\mathcal{C}^h = \mathcal{B} \cup D_1(\mathcal{B}) \cup D_2(\mathcal{B}) \cup \cdots \cup D_h(\mathcal{B}).$$

Lemma 6.1. *Under the case assumption, for* $0 \leq j \leq 3t-1$ *and* $0 \leq h \leq t$,

$$|\mathcal{C}^h[q+3t-j]| \geq \sum_{i=0}^{h} \mathcal{B}[q+3t-j+i](1-\delta)^i.$$

Proof. Note that the result holds for $j=0$ since $\mathcal{B}[q+3t+i]$ is empty for $i>0$. So assume $j \geq 1$ and proceed by induction on h. For $h=0$ the result is trivial since $\mathcal{C}^0 = \mathcal{B}$. For $h>0$,

$$\mathcal{C}^h[q+3t-j] = \mathcal{B}[q+3t-j] \cup D_1(\mathcal{C}^{h-1}[q+3t-j+1]).$$

By the induction hypothesis, since $j \geq 1$,

$$\begin{aligned} \mathcal{C}^{h-1}[q+3t-(j-1)] &\geq \sum_{i=0}^{h-1} \mathcal{B}[q+3t-(j-1)+i](1-\delta)^i \\ &= \sum_{i=1}^{h} \mathcal{B}[q+3t-j+i](1-\delta)^{i-1}. \end{aligned}$$

If this is non-zero then $|\mathcal{B}[q+3t-j+i]|$ is non-zero for some $i \leq \min(j,h)$ and by assumption is at least $|\mathcal{A}|/4t$. Thus $|\mathcal{C}^{h-1}[q+3t-(j-1)]| \geq (1-\delta)^{\min(j,h)-1}|\mathcal{A}|/4t$. Every element in $\mathcal{B}$ has down degree at least b (by hypothesis), so by Lemma 2.2, every element in $\mathcal{C}^h$ has down degree at least $b-t$. By the case assumption, $d_1(\mathcal{C}^{h-1}[q+3t-(j-1)]) \geq (1-\delta)\,|\mathcal{C}^{h-1}[q+3t-(j-1)]| \geq \sum_{i=1}^{h} \mathcal{B}[q+3t-j+i]\,(1-\delta)^i$. Since $\mathcal{B}$ is an antichain, $\mathcal{B}[q+3t-j]$ is disjoint from $D_1(\mathcal{C}^{h-1}[q+3t-(j-1)])$ and the lemma follows. □

Applying the lemma yields

$$\begin{aligned} |\mathcal{L}[q+1, q+4t]| \geq |\mathcal{C}^t[q+1, q+3t]| &\geq \sum_{j=0}^{3t-1} \sum_{i=0}^{t} \mathcal{B}[q+3t-j+i](1-\delta)^i \\ &= \sum_{i=0}^{t} (1-\delta)^i \sum_{j=0}^{3t-1} \mathcal{B}[q+3t-j+i] = \sum_{i=0}^{t} (1-\delta)^i\,|\mathcal{B}| \\ &\geq \frac{|\mathcal{A}|}{2} \sum_{i=0}^{t} (1-\delta)^i > R'\,|\mathcal{A}| \end{aligned}$$

to complete Case 2. □

This completes the proof of Theorem 1.1.

7. Upper bounds on $w(\mathscr{L})/|\mathscr{L}|$ and open problems

By examining the proof of Theorem 1.1 we can deduce an asymptotic upper bound on $w(\mathscr{L})/|\mathscr{L}|$. In Section 3 we showed $|w(\mathscr{L})| \geq n(R)$ implies $w(\mathscr{L})/|\mathscr{L}| \leq 1/R$. The function $n(R)$ was seen to satisfy $n(R) \leq 3(R'/c(b))^{1/\varepsilon(b)}$, where $R' = 3R$, b is obtained from Lemma 3.2 and $c(b)$ and $\varepsilon(b)$ are given by Lemma 3.1. Hence

$$n(R) \leq (k_1 R)^{2^b} \tag{7.1}$$

for some constant k_1. Taking $\delta = \frac{1}{4}R'$ in the proof of Lemma 3.2 we can take $t \leq k_2 R \log R$. Now b must be large enough so that $\theta(b-t, 2t) \leq \frac{1}{2}$, where θ is given in Lemma 5.1. Thus there exists a constant k_3 such that $b = k_3 R^2 (\log R)^3$ is large enough. From (7.1),

$$\log \log n(R) \leq k_4 R^2 (\log R)^3,$$

from which we can deduce

$$\frac{1}{R} \leq k_5 \sqrt[3]{\log \log \log n(R)} / \sqrt[2]{\log \log n(R)}$$

so we have

Theorem 7.1. *For a distributive lattice* $\mathscr{L}$

$$\frac{w(\mathscr{L})}{|\mathscr{L}|} = \mathrm{O}((\log \log |\mathscr{L}|)^{-(1/2-\varepsilon)})$$

for any constant ε.

As stated in the introduction, we expect that $w(\mathscr{L})/|\mathscr{L}| = 0((\log |\mathscr{L}|)^{-1/2})$. A way to come close to this bound would be to improve the lower bound for $\varepsilon(b)$ in Lemma 3.1 to a constant multiple of $1/b$. We believe this to be the correct bound. (To attain it let P be the union of b large chains of equal size and let $\mathscr{A}$ be the largest antichain of $\mathscr{L}(P)$.) Such a result would imply $w(\mathscr{L})/|\mathscr{L}| = \mathrm{O}(\log |\mathscr{L}|)^{-(1/2-\varepsilon)}$ for all ε.

The formulation in terms of finite sets (Theorem 1.2) suggests the following strengthening. For a collection $\mathscr{S}$ of finite sets let $H(\mathscr{S})$ be the set of all sets expressible as the union or intersection of exactly two members of $\mathscr{S}$.

Conjecture 7.2. *For every number* $R > 0$, *there exists a number* $n(R)$ *such that if* $\mathscr{S}$ *is an antichain of finite sets and* $|\mathscr{S}| \geq n(R)$, *then* $|H(\mathscr{S})| \geq R\,|\mathscr{S}|$.

The reader may note a similarity to a theorem of Daykin [4] which says that if $\mathscr{A}$ is a collection of sets and $\mathscr{A} \vee \mathscr{A}$ (resp. $\mathscr{A} \wedge \mathscr{A}$) is the set of all sets expressible as $A \cup B$ (resp. $A \cap B$) for $A, B \in \mathscr{A}$, then $|\mathscr{A} \vee \mathscr{A}|\,|\mathscr{A} \wedge \mathscr{A}| \geq \mathscr{A}^2$. Daykin's result is in fact best possible; the thrust of the present conjecture is that something stronger can be said when $\mathscr{A}$ is an antichain of sets.

One interesting problem that arises from a consideration of a special case of Lemma 5.1 is: how are the sizes of consecutive levels of a distributive lattice related? The techniques of Lemma 5.1 can be used to show

Theorem 7.3. $|\mathscr{L}[i+1]|\,|\mathscr{L}[i-1]| \geqslant \frac{1}{4}\,|\mathscr{L}[i]|^2$.

We believe that this can be improved:

Conjecture 7.4. $|\mathscr{L}[i+1]|\,|\mathscr{L}[i-1]| \geqslant \binom{|\mathscr{L}[i]|}{2}$.

If true, this is best possible since equality holds for the first three ranks of a Boolean Algebra.

A related question concerns the average down degree of the elements of a rank. By Lemma 2.2 we know that if Y covers X, then $1 + d_1(X) \geqslant d_1(Y)$. An intriguing question is whether this inequality can be averaged over ranks:

Question 7.5. Is it the case that for any $i \geqslant 1$,

$$1 + \frac{1}{|\mathscr{L}[i]|} \sum_{X \in \mathscr{L}[i]} d_1(X) \geqslant \frac{1}{|\mathscr{L}[i+1]|} \sum_{Y \in \mathscr{L}[i+1]} d_1(Y)?$$

Finally, it is natural to ask whether the analog of Theorem 1.1 holds for other classes of lattices. It does not hold for all modular lattices since the lattice consisting of an antichain together with a minimum and maximum element is modular. One class of lattices for which we believe it holds is meet distributive lattices. These are lattices satisfying the property: if $X \in \mathscr{L}$ and Y is the meet of elements covered by X then the interval $[Y, X]$ is isomorphic to a Boolean Algebra. This class, which includes distributive lattices, has been studied extensively (see [5, 6]).

Conjecture 7.6. $\lim_{|\mathscr{L}|\to\infty} w(\mathscr{L})/|\mathscr{L}| \to 0$, *where* $\mathscr{L}$ *ranges over meet distributive lattices.*

Almost all of the arguments in this paper carry over to meet distributive lattices. For instance Lemma 3.1 has an analog via the representation theorem for meet distributive lattices as the lattice of closed sets of an anti exchange closure [6]. However, the proof of Lemma 5.1 does not carry over to meet distributive lattices, and indeed we do not know whether Lemma 5.1 holds here. A key difficulty is that meet distributive lattices are not necessarily modular. Nevertheless, we believe the conjecture is true.

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